Vector Calculus: Tensors

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Here are some supplementary notes for the section on Tensors in the *Vector Calculus* course from Part IA of the Mathematical Tripos.

We work in three dimensions, although the ideas generalise straightly. The summation convention applies throughout unless otherwise stated.

1 Revision: Vectors and coordinates

Recall that a *vector* $\mathbf{v}$ is a quantity that has a magnitude and a direction, such as a displacement, a velocity or a force. Given an orthonormal basis $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$, we can write

$$\mathbf{v} = v_i \mathbf{e}_i$$

for some unique numbers v_i, which are *coordinates* of $\mathbf{v}$. The vector itself is a geometric or physical quantity, the coordinates are numbers which describe the vector relative to a basis.

Now consider another orthonormal basis $\mathbf{e}'_i$. Since both bases are orthonormal, there is an orthogonal matrix R such that

$$\mathbf{e}'_i = R_{ji} \mathbf{e}_j.$$

Hence, if we have

$$\mathbf{v} = v_i \mathbf{e}_i = v'_i \mathbf{e}'_i$$

then, as usual, we have

$$v_i = R_{ij} v'_j.$$

Since R is orthogonal, we have $R^{-1} = R^T$, so

$$(R^T R)_{ij} = R_{ki} R_{kj} = \delta_{ij}$$

and

$$(R R^T)_{ij} = R_{ik} R_{jk} = \delta_{ij}.$$

Hence we also have

$$v'_i = R_{ji} v_j,$$

the *tensor transformation rule.*

Hence for a fixed vector and different orthonormal bases related by R, the coordinates of the vector with respect to the different axes must satisfy this equation.

Vectors in three dimensions have three coordinates, and a coordinate can be specified by *one* suffix: v_1, v_2, v_3. Thus, a vector is a *rank one* tensor.

2 Scalars

A *scalar* quantity x is a quantity described by a single number, which does not change under change of bases. (The mass of a particle does not change if you look at it from another direction.) Since a scalar has a single number, *no* suffices are required to specify this number: so a scalar is a *rank zero* tensor. The transformation rule is simply $x' = x$.

3 Example of a tensor: Electrical conductivity

In electromagnetism, an electric field $\mathbf{E}$ in a conducting material will cause a current $\mathbf{J}$ to flow in the material. The electric field and the current are linearly related, i.e. there is a linear map σ with

$$\mathbf{J} = \sigma \mathbf{E}.$$

The linear map σ is called the *conductivity tensor* of the material. On picking an orthonormal basis $\mathbf{e}_i$, we have

$$J_i = \sigma_{ij} E_j$$

where (σ_{ij}) is the matrix of the linear map with respect to this basis.

As coordinates are to vectors, so matrices are to linear maps. Elements of matrices can be specified by *two* suffices, and so linear maps are *rank two* tensors.

Recall from *Vectors and Matrices*: If $\mathbf{e}'_i$ is another orthonormal basis with $\mathbf{e}'_i = R_{ji}\mathbf{e}_j$, and with respect to this new basis σ has matrix (σ'_{ij}), then

$$\sigma' = R^{-1}\sigma R = R^T \sigma R.$$

We then have

$$\sigma'_{ij} = R_{ki} R_{lj} \sigma_{kl}$$

so, like coordinates of vectors, the matrices of σ with respect to the different bases are also related by a tensor transformation rule.

4 Isotropic tensors

In general, the vector quantities $\mathbf{E}$ and $\mathbf{J}$ may be linearly related without being directly proportional. However, if σ is a scalar map (a scalar multiple of the identity, so $\sigma_{ij} = \delta_{ij}$, with respect to any orthonormal basis) then the material is called an *isotropic* conductor. For such a material, the electric field and current are directly proportional. This means that the conductor behaves the same no matter from which direction we look at it.

An example of an *anisotropic* electrical conductor is graphite, which consists of stacked sheets of carbon atoms with delocalised electrons. An electric field will create a larger current if it is parallel to the sheets than if it is perpendicular. Graphite is also an anisotropic conductor of heat, since excitations travel slowly between sheets but quickly on a given sheet. The conductivity tensor of graphite might look something like

$$\begin{pmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & b \end{pmatrix}$$

for some $a > b$.

Isotropic tensors of rank 0, 1, 2, 3 Rank 0 tensors are scalars, so they are always isotropic – a scalar quantity such as temperature has no direction, and it doesn't look different when you look at it from another direction.

Rank 1 tensors are vectors, and obviously a vector looks different when you look at it from another direction, so it cannot be isotropic. The only excetion to this is the zero vector.

It can be shown that the only isotropic rank 2 tensors are δ_{ij} and its scalar multiples. Remember that rank 2 tensors are linear maps. This therefore is the assertion that the only linear map whose effects don't look any different under any rotations are identity map and its scalar multiples (enlargements about the origin).

It can also be shown that the only isotropic rank 3 tensors are multiples of ϵ_{ijk}.

5 (*) A more abstract way of thinking about tensors

A more abstract way of thinking about tensors is to think about their algebraic properties. How do they interact with other mathematical objects?

Suppose we have a vector space V over a field F. Recall that the most general way of defining a linear map $V \to F$ is to take some vector v, and let our linear map be 'dot with v'.