Topographical effects on granular gravity currents

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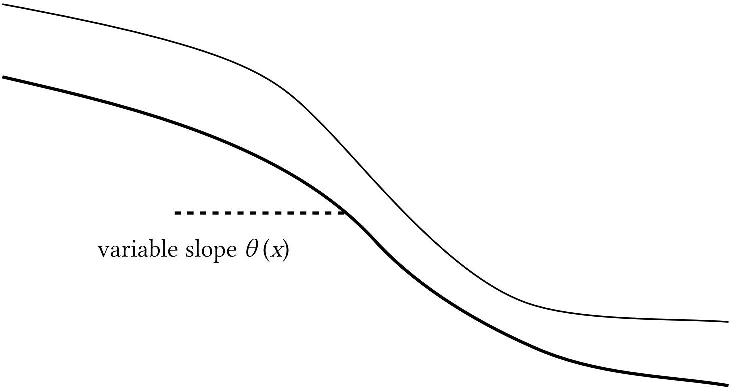
Why study this?

- Mathematically interesting
  - Equations of motion have similar structure to hydrodynamics
  - **Classical problems, e.g. Blasius boundary layers**
- As a toybox for understanding granular rheology
  - **Can have all three 'states' of granular matter**
- Practical uses
  - Landslides and avalanches
  - Topography represents hills, defences or houses

Methods

- **$\mu(I)$ rheology (e.g. Jop et al. 2005)**
  - **Depth-averaged models ('shallow sand')**
  - **Boundary conditions**
- Other continuum models
  - Granular gas model (*e.g.* Haff 1983)
  - 'Solid state' models
- Discrete particle model (DPM) simulations
  - Using MercuryDPM (developed at Univ. Twente, NL)
  - Can look at velocity inside a current, unlike lab experiments

Depth-averaged model



Depth-averaged model

- From Gray and Edwards 2014: Depth-averaged $\mu(I)$ equations

$$h_t + (h\overline{u})_x = 0$$

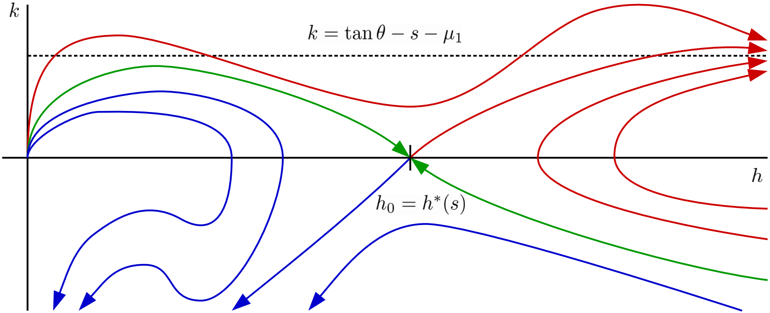
$$(h\overline{u})_t + (\chi h\overline{u}^2)_x + \left(\frac{1}{2}gh^2\cos\theta\right)_x = ghS + \left(\nu h^{3/2}\overline{u}_x\right)_x.$$

- For steady flow, $\dfrac{\partial}{\partial t} = 0$ and $h\overline{u} = Q$ is constant:

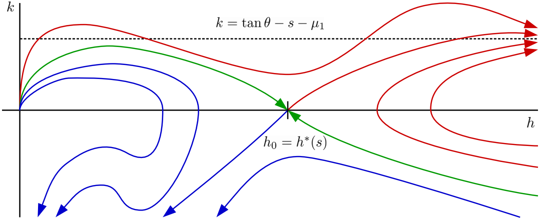
$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\chi Q^2}{h} + \frac{1}{2}gh^2\cos\theta + \frac{Q\nu}{h^{1/2}}\frac{\mathrm{d}h}{\mathrm{d}x}\right) = g\cos\theta\left(\tan\theta - M(h) - \frac{\mathrm{d}b}{\mathrm{d}x}\right)h.$$

Depth-averaged model

- $\theta =$ constant $\implies$ h governed by autonomous 2^{nd}-order ODE
- Phase-plane analysis



Depth-averaged model



- For constant θ, h **equilibriates** to Bagnold depth $h_{\text{Bag}}(Q, \theta)$
  - Otherwise, $h \to 0$ or $h \to \infty$ as $x \to \infty$, violating BCs
- At equilibrium depth, $\mu(I) = \tan\theta$
- Lower $\theta \implies$ higher h_{Bag} (lower speed, lower I, lower $\mu(I)$)

Boundary layer

- If θ changes then h must **adjust** towards new h_{Bag}

$\therefore$ **Deviation from Bagnold profile**

- Consider $h > h_{\text{Bag}}(Q, \theta)$: flow is 'too fast' for the new slope
- Need to shift otherwise Bagnoldian profile to make depth and flow rate match
- Need **boundary layer** to impose **no-slip condition** at base
  - No-slip applies provided that the base is rough at grain-scale to prevent sliding or rolling
  - Doesn't apply for glass spheres over flat glass plate

Boundary layer equations

- Assume $\eta = \dfrac{\text{depth of current}}{\text{topographical lengthscale}} \ll 1$ for shallow flow
- Consider supercritical flow $\mathcal{F} \gg \eta^{-1/2}$
- Have (after nondimensionalisation and dropping $O(\eta)$ terms)

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0, \qquad \epsilon\eta\mathcal{F}^2 u^2 \frac{\mathrm{d}\theta}{\mathrm{d}x} = \frac{\partial p}{\partial z},$$

$$\epsilon\eta\mathcal{F}^2 \left(u \frac{\partial u}{\partial x} + w \frac{\partial u}{\partial z} \right) = \frac{\partial}{\partial z} \left(\mu(I) p \right)$$

- Have distinguished scaling $\epsilon = \dfrac{\text{BL thickness}}{\text{depth of current}} \sim (\eta\mathcal{F}^2)^{-1}$

Boundary layer equations

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0, \qquad \epsilon\eta\mathcal{F}^2 u^2 \frac{\mathrm{d}\theta}{\mathrm{d}x} = \frac{\partial p}{\partial z},$$

$$\epsilon\eta\mathcal{F}^2 \left(u \frac{\partial u}{\partial x} + w \frac{\partial u}{\partial z} \right) = \frac{\partial}{\partial z} \left(\mu(I) p \right)$$

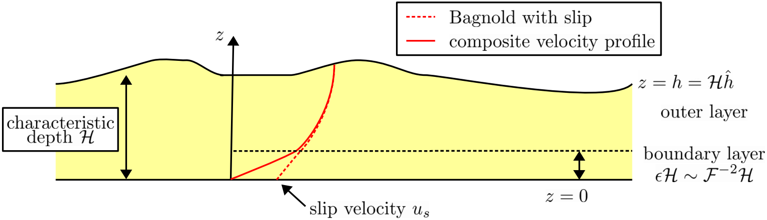
- $I \propto p^{-1/2} \dfrac{\partial u}{\partial z}$ so BL equation is 2^{nd}-order
- Similar structure to Prandtl BL equations (PBLE):

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0, \qquad \frac{\partial p}{\partial z} = 0,$$

$$\epsilon^2 \mathrm{Re} \left(u \frac{\partial u}{\partial x} + w \frac{\partial u}{\partial z} \right) = -\frac{\mathrm{d}p}{\mathrm{d}x} + \frac{\partial^2 u}{\partial z^2}$$

Boundary layer equations

- **Assuming** that solutions behave like those of PBLE, can **match** BL velocity profile to outer layer



Challenge I: Analysing the BL equations

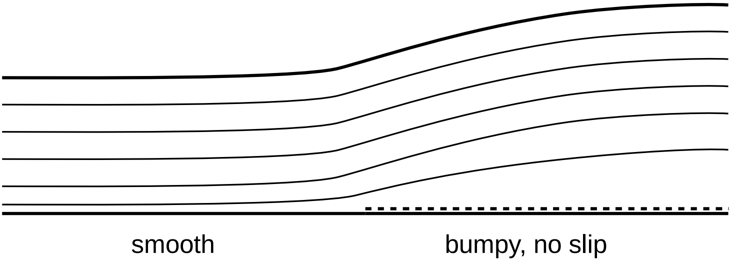
$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0, \qquad \epsilon\eta\mathcal{F}^2 u^2 \frac{\mathrm{d}\theta}{\mathrm{d}x} = \frac{\partial p}{\partial z},$$

$$\epsilon\eta\mathcal{F}^2 \left(u \frac{\partial u}{\partial x} + w \frac{\partial u}{\partial z} \right) = \frac{\partial}{\partial z} \left(\mu(I) p \right)$$

- **Nonlinear PDEs are hard(!)**
  - Prandtl BL equations are at least quasilinear
  - $\mu(I)$ BL equations are fully nonlinear
  - **Existence, uniqueness and stability of solutions?**
  - **Numerical methods?**

Challenge II: Matching theory to experiment

- **How can we test the scaling law $\epsilon \sim (\eta \mathcal{F}^2)^{-1}$?**
  - Generate flows over topography in DPM simulations (or lab)
  - Rough surface for $x > 0$ analogous to classical Blasius problem
  - Measure velocity profiles



Challenge II: Matching theory to experiment

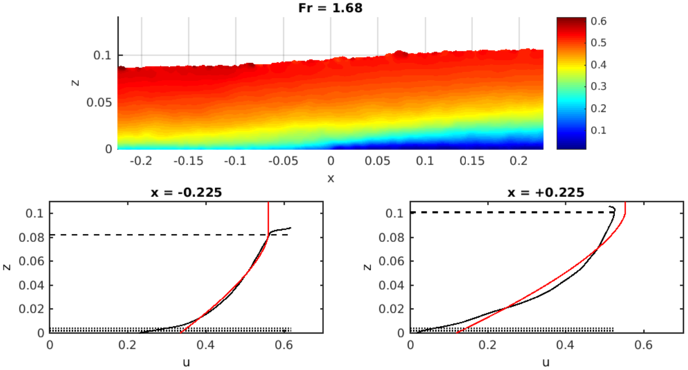


Figure: DPM simulation of flow over a smooth surface for $x < 0$ incident onto a rough surface for $x > 0$, analogous to Blasius plate problem. Black: Coarse-grained velocity profiles; Red: fitted Bagnold+slip profiles.

Challenge II: Matching theory to experiment

- DPM simulations show deviation from Bagnold when no-slip condition is suddenly applied
- **How to determine empirical BL thickness from given velocity profile?**
- $\epsilon \sim (\eta \mathcal{F}^2)^{-1}$ has unknown constant of proportionality

Summary

- Velocity profile is Bagnoldian $\Leftrightarrow h = h_{\text{Bag}}(Q, \theta)$
- For $h \neq h_{\text{Bag}}$, postulate **boundary layer**
  - Get Prandtl-like BL equations, but much more nonlinear
  - Changing friction or slope similar to pressure gradient
  - Adverse pressure gradient may cause BL separation
- **Challenges**
  - Analysis of BL equations for $h < h_{\text{Bag}}$
  - Comparing with empirical results