Dimensional analysis, scaling laws and asymptotics

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Introduction

Physical systems can be very complicated. The equations that describe them are often difficult or impossible to solve exactly. We usually have to make assumptions or approximations about the system.

The simple observation that you can't add different types of quantities leads to a powerful tool, *dimensional analysis*. Dimensional analysis doesn't give exact answers, but it can tell you how the different quantities are related to each other..

Asymptotic analysis is the study of how small changes to a problem affect the solution. We often assume that small differences between problems should result in small changes in the solutions. This is sometimes true.

Dimensions

- Most physical quantities come with *dimensions*: ‘length’ (L), ‘time’ (T), ‘velocity’ (LT^{-1}).
- You can’t add or subtract physical quantities with different dimensions:
  - distance from Earth to Sun $+$ time until lunch $=$?
  - price of oil $+$ wealth of Donald Trump $=$?
- You can multiply and divide different quantities:
  - speed $= \frac{\text{distance}}{\text{time}}$
  - How many barrels of oil can Donald Trump buy?
- Dimensions and units are related concepts, but dimensions are more abstract: e.g. $1\mathrm{m} + 1\mathrm{ft}$ is legal.

Dimensions

- Some quantities are *dimensionless*: e.g. aspect ratios, angles (ratios of lengths), the number of people in a room, probabilities.
- Most mathematical functions, such as sin and cos, can only be applied to dimensionless quantities. For example, you can't apply sin to a length:

$$\sin(x) = x - x^3/6 + x^5/120 - \cdots$$

  and so, if we applied sin to a dimensional quantity, we would have

$$\sin(2\mathrm{m}) = 2\mathrm{m} - \frac{4}{3}\mathrm{m}^3 + \cdots$$

  which is illegal.
- This is related to the fact that you should be able to change the unit of measurement for lengths without changing the answer.

Dimensional analysis

- Physical laws must be dimensionally consistent and can't depend on the units that you express them in. This fact can be used to simplify problems if we have a sensible idea of what the answer can depend on.
- Laws can be expressed as relationships between *dimensionless groups* and constant numbers: e.g. as a relationship of the form $f(x_1, x_2, \dots) = c$.
- For example:

$$F = ma \text{ can be written as } \frac{F}{ma} = 1,$$

and

$$s = ut + \frac{1}{2}at^2 \text{ as } \frac{s}{ut} = 1 + \frac{at}{2u}.$$

Dimensional analysis

- What is the period T of a pendulum of length L and mass m undergoing small oscillations (ignoring air resistance)?
- The quantities in play are T, the mass of the pendulum m, its length L, and the strength of gravity g.
- Now T has dimensions of time, m, of mass, L, of length, and g, of length $\times$ (time)$^{-2}$.
- The only possible dimensionless group is $T/\sqrt{L/g}$.
- Hence $T/\sqrt{L/g}$ must be equal to a (dimensionless) constant C.
- So, $T = C\sqrt{L/g}$, and independent of m.

Why dimensional analysis?

- The pendulum's angle $\theta(t)$ is governed by the differential equation
$$mL\ddot{\theta}(t) = -mg\theta(t)$$
and we could have solved this to get that
$$\theta(t) = \theta_0 \cos\left(\sqrt{\frac{g}{L}}(t - t_0)\right)$$
which tells us that the pendulum's period is $T = 2\pi\sqrt{L/g}$.
(Don't worry if you don't know how to do this yet.)

Why dimensional analysis?

- Dimensional analysis didn't tell us about the 2π.
- But it did tell us that $T = C\sqrt{L/g}$ for *some* number C, and that T doesn't depend on m at all. We say that T *scales like* $(L/g)^{1/2}$.
- If we can determine C for one pendulum, then we we will know T for all pendula, whatever their lengths and masses (assuming that the oscillations are small). This is called *dynamical similarity*.
- In this example, we could have solved the differential equation exactly to get T exactly. But for other, more complicated systems, it may not be possible to solve the DE, or even to set it up!

Fluid dynamics: The flow past a sphere

- Consider a body of fluid at rest, and suppose there is a sphere of radius a immersed in it. The sphere has the same density as the fluid, so that it neither sinks nor rises. Let the sphere move at a constant speed U. What is the drag force F on the sphere?
- The answer depends on the values of U and a as well as the nature of the fluid: specifically, on the fluid's *dynamic viscosity*, μ, and its density, ρ.
- The dynamic viscosity is a measure of how much force is needed to drive the fluid at a given flow rate; it has dimensions of density $\times$ area $\times$ (time)$^{-1}$, or equivalently, mass $\times$ (length)$^{-1}$ $\times$ (time)$^{-1}$.

Fluid dynamics: Dimensional analysis

- We want to construct a force F, which has dimensions of mass $\times$ length $\times$ (time)$^{-2}$. There are four parameters to play with: U, a, μ and ρ.

- It can be shown that there are two dimensionless groups:

$$\frac{F}{\mu Ua} \text{ and } \frac{\rho Ua}{\mu}.$$

  (You can construct other dimensionless groups, but they will simply be rearrangements of these two.)

- Hence, any physical law of this system must be of the form

$$\frac{F}{\mu Ua} = f\left(\frac{\rho Ua}{\mu}\right)$$

  for some function f.

- Hence,

$$F = \mu Uaf\left(\frac{\rho Ua}{\mu}\right).$$

Fluid dynamics: The Reynolds number

$$F = \mu U a f\left(\frac{\rho U a}{\mu}\right)$$

- The dimensionless combination $\rho U a/\mu$ has a special name: the *Reynolds number*, Re. It measures how fast the sphere is moving and how large it is, compared to the viscosity of the fluid.
- Now we 'just' need to determine the function f.

Fluid dynamics: Drag coefficients

$$F = \mu U a f\left(\frac{\rho U a}{\mu}\right)$$

- We can't determine f exactly because the equations governing fluid flow (the Navier-Stokes equations) cannot be solved exactly, although it can be shown that $f(\mathrm{Re}) \approx 6\pi$ for small values of Re.
- It is relatively straightforward to determine f experimentally or using computer simulations. These have shown that $f(\mathrm{Re}) \approx c_D \mathrm{Re}$ as $\mathrm{Re} \to \infty$. The constant c_D is sometimes called the *drag coefficient* of a sphere.
- However, there is a snag with taking $\mathrm{Re} \to \infty$... see later.

Fluid dynamics: Drag coefficients

- To summarise:

$$F = \mu U a f(\mathrm{Re}) \text{ where } \mathrm{Re} = \rho U a/\mu,$$

  and

$$f(\mathrm{Re}) \sim \mathrm{Re} \text{ as } \mathrm{Re} \to \infty.$$

- Thus

$$F \sim \rho U^2 a^2 c_D$$

  when the Reynolds number is large, i.e. when the sphere is travelling very fast.

- Other shapes have different functions f and different drag coefficients. (For a sphere, a is the only lengthscale. With other shapes, there will be more lengthscales, and aspect ratios will be extra nondimensional quantities.)

Fluid dynamics: The Reynolds number

$$F = \mu U a f(\mathrm{Re}) \text{ where } \mathrm{Re} = \rho U a/\mu$$

- Instead of exploring four different parameters (μ, ρ, U and a), we now just to explore one parameter, the combination Re.
- If we know the value of $f(\mathrm{Re})$ at a certain value of Re, then we can work out the drag force on any sphere of any size moving at any speed in any Newtonian fluid, moving with that Reynolds number. A large sphere in oil behaves like a small sphere in water.
- This is another example of dynamical similarity.
- For a flow past a sphere, the Reynolds number alone determines the nature of the whole flow.

Fluid dynamics: The Reynolds number

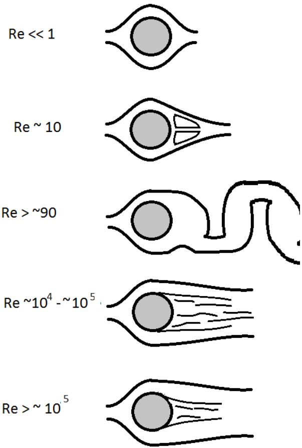

Figure: Source: Wikipedia,
https://en.wikipedia.org/wiki/File:Reynolds_behaviors.png

Fluid dynamics: The Reynolds number

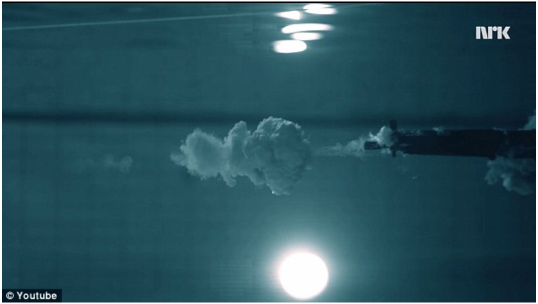

Figure: A bullet moving through water has a Reynolds number of around 10^7. The Reynolds number is even higher in air. At such high Re, the wake of the bullet is extremely turbulent. Source: Andreas Wahl

Fluid dynamics: Dynamical similarity

- Fish and tadpoles beat their tails back and forth, but sperm cells spin their tails like a corkscrew.
- Microorganisms and sperm cells are very small, and swim in a low-Reynolds number environment ($\mathrm{Re} \approx 10^{-5}$). Tadpoles and fish swim with much higher Reynolds numbers ($\mathrm{Re} \approx 300$ for a tadpole, $\mathrm{Re} \approx 40000$ for a large trout).
- Their swimming styles are adapted to the fluid dynamics of their environment. A tadpole would not be able to swim if it were scaled down. A sperm cell would not be able to swim if it were scaled up.
- But a spinning corkscrew immersed in syrup (about 1000 times more viscous than water) would be able to move. It is much larger than a sperm cell, but also in a much more viscous fluid, so they have a similar Reynolds number.

Dimensional analysis: Summary

The two examples of a pendulum and a translating sphere are particularly simple, but they illustrate the method of dimensional analysis:

First, identify the things that the answer can possibly depend on. Then, construct dimensionless groups out of them. The dimensionless groups are then related to each other by some function.

In practice, the first step is often the hardest, because it requires us to make modelling assumptions about whatever physical system we are working with.

Asymptotic analysis

In the last part of this talk, we look at examples where small changes to a problem can lead either to a small change or to a large change in the solution, depending on the problem. To keep things simple, our examples will be mathematical and abstract rather than physical.

Asymptotic analysis: Small effect, small change

- First, consider the quadratic equation

$$x^2 - (3+\epsilon)x + 2 = 0.$$

  where ϵ is a small number.

- When $\epsilon = 0$, the two solutions are $x = 1$ and $x = 2$.
- In general, the solutions are given by

$$x_{\pm} = \frac{1}{2}\left(3 + \epsilon \pm \sqrt{(3+\epsilon)^2 - 8}\right).$$

Asymptotic analysis: Small effect, small change

$$x^2 - (3+\epsilon)x + 2 = 0$$

$$x_\pm = \frac{1}{2}\left(3 + \epsilon \pm \sqrt{(3+\epsilon)^2 - 8}\right)$$

- Using $(a+b)^{1/2} \approx a^{1/2} + \frac{1}{2}a^{-1/2}b$, we have

  $$x_\pm \approx \frac{1}{2}\left(3 + \epsilon \pm (1 + 3\epsilon)\right)$$

  and so

  $$x_+ \approx 2 + 2\epsilon \text{ and } x_- \approx 1 - \epsilon.$$

- So when one of the coefficients in the quadratic equation is changed by a small amount ϵ, the solutions change by amounts proportional to ϵ.

Asymptotic analysis: Small effect, large change

- Now consider the quadratic equation

$$\epsilon x^2 - 2x + 1 = 0.$$

- For $\epsilon = 0$, there is only one solution, $x = 1/2$, but for all other values of ϵ there are two. What are the solutions like when ϵ is small?

- The exact solutions are

$$x_{\pm} = \frac{1}{\epsilon}\left(1 \pm \sqrt{1-\epsilon}\right)$$

and we can write

$$x_{\pm} \approx \frac{1}{\epsilon}\left(1 \pm (1 - \epsilon/2)\right).$$

Hence $x_{-} \approx 1/2$ and $x_{+} \approx 2/\epsilon$.

Asymptotic analysis: Small effect, large change

- The equation $\epsilon x^2 - 2x + 1$ has very different behaviour between the cases $\epsilon = 0$ and ϵ small but nonzero. When ϵ is small, then small changes to ϵ cause very large changes to x_+, and x_+ goes to infinity as ϵ goes to zero.
- So although ϵ may be arbitrarily small, we cannot get accurate solutions if we ignore it completely.
- This is an example of a *singular perturbation problem*.
- Singular perturbations problems are rather common in fluid mechanics and other branches of applied maths. They usually involve an equation like this one, where the 'most important' term (e.g. highest power or highest-order derivative) is multiplied by the small parameter.

Asymptotic analysis: Small effect, large change

- It turns out that the limit $\mathrm{Re} \to \infty$ is a singular limit.
- A *perfectly inviscid* fluid (such as liquid helium) flowing past an object behaves very differently from a *low viscosity* fluid (such as air).
- For example, when a ball flies through the air, it leaves a turbulent wake. Liquid helium simply creeps around the ball.
- A fish in liquid helium would not be able to swim.

Fin.

Thank you for your attention!

These slides are available at
`http://jmft2.user.srcf.net/dasla.pdf`.